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CONSERVATION LAWS FOR NONHOMOGENEOUS BERNOULLI–EULER BEAMS

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Abstract—It is the purpose of this paper to construct conservation laws for the statics and dynamics of nonhomogeneous Bernoulli–Euler beams. To derive these conservation laws, we will use the newly proposed Neutral Action (NA) method (Honein *et al.*, 1991, *Phys. Lett.*, **155**, 223–224; Chien, 1992, Conservation laws in nonhomogeneous and dissipative mechanical systems, Ph.D. Dissertation, Stanford University). The conservation laws derived should be useful in characterizing concentrated defects, such as cracks and interfaces, in an otherwise smoothly nonhomogeneous beam.

Classically, Noether's first theorem (Noether, 1918, *Transport Theory Stat. Phys.* **1**, 186–207) is available for construction of conservation laws for Lagrangian systems, such as a Bernoulli–Euler beam. However, since the NA method is applicable to dissipative as well as to Lagrangian systems, and since it encompasses Noether's method within the realm of Lagrangian systems, we choose to employ the NA method to achieve our purpose here. A comparison of these two methodologies, with an example illustrating the relative efficiency of the NA method over Noether's approach, will also be presented.

1. INTRODUCTION

Since the introduction of the J, L and M integrals in elastic solids, the importance of path-independent integrals has been widely recognized. These path-independent integrals are representations, in divergence-free form, of conservation laws in material space. Classically, for Lagrangian systems, conservation laws can be obtained via Noether's first theorem (1918). Details of this classical method can be found, for example, in Logan (1977), Olver (1986) and Bluman and Kumei (1989). Until recently, Noether's methodology was the only systematic approach available for construction of conservation laws, but it is valid only for nondissipative systems. However, in a recent brief note entitled "On Conservation Laws for Dissipative Systems" (Honein *et al.*, 1991), a new approach for constructing conservation laws was proposed. It was termed the "Neutral Action (NA)" method in Chien (1992). Given any system governed by a set of differential equations, the NA method allows one to systematically obtain conservation laws that are valid for the system considered. Since the basic building block for conservation laws is the governing differential equations, this new method can be applied to Lagrangian systems as well as to dissipative systems without a Lagrangian, i.e. to any set of differential equations, regardless of whether they are the Euler–Lagrange equations of a variational problem or not.

It is the purpose of this present contribution to derive conservation laws for the statics and dynamics of a nonhomogeneous Bernoulli–Euler beam using the NA method. Also, the NA method will be compared to the classical Noether's first theorem within the context of Lagrangian systems.

The conservation laws presented here are not exhaustive, but as a limited set, they should provide valuable insights into the behavior of non-homogeneous beams. A typical use of these laws is in the analysis of defect mechanics, such as jump-discontinuities and cracks (Kienzler and Herrmann, 1986b) in an otherwise smoothly nonhomogeneous beam.

2. NEUTRAL ACTION (NA) METHOD FOR CONSTRUCTING CONSERVATION LAWS

Given any system with m independent variables x^i $(i = 1, 2, \ldots, m)$, n dependent variables u^k $(k = 1, 2, \ldots, n)$, the governing set of q equations can be represented by:

$$\Delta^\alpha(x^i, u^k, u^k_J) = 0 \quad (\alpha = 1, 2, \ldots, q). \tag{1}$$

Using the multi-index notation introduced by Olver (1986), u^k_J in eqn (1) represents all possible pth order partial derivatives of u^k,

$$u^k_J \equiv \frac{\partial^p u^k}{\partial x^{j1} \partial x^{j2} \ldots \partial x^{jp}},$$

with $J = (j_1, j_2, \ldots, j_p)$ as an unordered p-tuple of integers, $1 \leqslant j_\alpha \leqslant m$ indicating which derivatives are being taken, and $\# J = p$ indicating how many derivatives are being taken.

For any system governed by a set of differential equations as in eqn (1), the "Neutral Action (NA)" method proposed by Honein *et al.* (1991) states that it is possible to construct conservation laws valid for the system in the form:

$$f^\alpha \Delta^\alpha = D_i P^i = 0, \tag{2}$$

if

$$E^k(f^\alpha \Delta^\alpha) \equiv 0, \tag{3}$$

for all dependent variables u^k. Here, $f^\alpha = f^\alpha(x^i, u^k, u^k_J)$ are the unknown characteristics of conservation laws, $P^i = P^i(x^i, u^k, u^k_J)$ are the conserved currents of the balance law $D_i P^i = 0$, and E^k is the Euler operator defined as:

$$E^k(L) = (-D)_J \frac{\partial L}{\partial u^k_J}, \quad 0 \leqslant \# J \leqslant p, \tag{4}$$

with

$$D_J \equiv D_{j1} D_{j2} \ldots D_{jp},$$

representing all possible pth order total derivatives, and

$$(-D)_J = (-1)^{\# J} D_J.$$

Also, summation over dummy indices is assumed throughout this paper.

Since our objective is to construct some divergence-free expressions out of $f^\alpha \Delta^\alpha$, and since the Euler operator acting on any total divergence always yields a null result by the calculus of variations, it follows that eqn (3) is a requirement for the existence of conservation laws. Equation (3) also implies that $f^\alpha \Delta^\alpha$ is a null Lagrangian whose action integral,

$$A = \int_\Omega f^\alpha \Delta^\alpha \, \mathrm{d}V, \tag{5}$$

has vanishing variation for any dependent variable u^k, i.e. $\delta A \equiv 0$. In other words, in order to construct conservation laws for any system (dissipative or Lagrangian) governed by a set of differential equations, $\Delta^\alpha = 0$, by the NA method, we try to construct a product of $f^\alpha \Delta^\alpha$ whose action integral does not change variationally. Hence the name "Neutral Action (NA)" method given to this procedure.

In practice, given any set of differential equations, $\Delta^\alpha = 0$, one must first decide on the arguments of the characteristics f^α, which may be functions of the independent variables, the dependent variables and some or all derivatives of the dependent variables. Subsequently, one would use eqn (3) to solve for these characteristics. Having solved for

f^α, one can then proceed to construct the conserved currents P^i valid for the system governed by this set of differential equations from the product $f^\alpha \Delta^\alpha$.

3. ELASTOSTATICS OF BERNOULLI–EULER BEAMS

3.1. *Beams with end loading*

The Lagrangian density of a nonhomogeneous beam subjected to end loading with material coordinate x, transverse displacement $u = u(x)$, and bending stiffness $B = B(x)$ is given by:

$$L = -\tfrac{1}{2}B(x)u_{xx}^2. \tag{6}$$

The associated Euler–Lagrange equation that governs this system is given by:

$$\Delta = E(L) = B_{xx}u_{xx} + 2B_x u_{xxx} + Bu_{xxxx} = 0. \tag{7}$$

In this paper, subscripts indicate partial differentiation.

Given the above governing differential equation of the system, one could use the method outlined in Section 2 to obtain the conservation laws valid for this system. That is, one needs to solve eqn (3),

$$E(f\Delta) \equiv 0, \tag{8}$$

for f, where the order of derivatives on which f may depend must be fixed *a priori*.

From eqn (7), any fourth or higher order derivatives of u can be expressed in terms of x, u_{xx} and u_{xxx}. Thus, for completeness of conservation laws within the framework of the NA method, one should require that the characteristic be of the form $f = f(x, u, u_x, u_{xx}, u_{xxx})$. However, on constructing the explicit form of eqn (8), one would need to calculate the fourth order total derivative of f. Using the above f would lead to extremely lengthy calculations and thus will not be pursued. Here, we will assume a characteristic f such that:

$$f = f(x, u, u_x). \tag{9}$$

Given the governing equation of the system, eqn (7), and having assumed the dependence of the characteristic f, the condition for the existence of conservation laws by the NA method, eqn (8), can be expanded as:

$$\frac{\partial(f\Delta)}{\partial u} - D_x\frac{\partial(f\Delta)}{\partial u_x} + D_{xx}\frac{\partial(f\Delta)}{\partial u_{xx}} - D_{xxx}\frac{\partial(f\Delta)}{\partial u_{xxx}} + D_{xxxx}\frac{\partial(f\Delta)}{\partial u_{xxxx}} = 0. \tag{10}$$

Since the only unknown in eqn (10) is the characteristic f which depends on x, u, and u_x, it follows that all coefficients of second and higher order derivatives of u in this equation must be set equal to zero independently. The resulting set of equations is as follows:

Coefficient	Equation
u_{xx}	$B_{xxx}\frac{\partial f}{\partial u_x} - B_{xx}\left(2\frac{\partial f}{\partial u} + \frac{\partial^2 f}{\partial x \partial u_x}\right) - B_x\left(6\frac{\partial^2 f}{\partial x \partial u} + 6\frac{\partial^3 f}{\partial x^2 \partial u_x}\right) - B\left(6\frac{\partial^3 f}{\partial x^2 \partial u} + 4\frac{\partial^4 f}{\partial x^3 \partial u_x}\right) = 0$
u_{xx}^2	$-3B\frac{\partial^2 f}{\partial u^2} = 0$
u_{xxx}	$2B_{xx}\frac{\partial f}{\partial u_x} - B_x\left(4\frac{\partial f}{\partial u} + 4\frac{\partial^2 f}{\partial x \partial u_x}\right) - B\left(4u_x\frac{\partial^2 f}{\partial u^2} + 4\frac{\partial^2 f}{\partial x \partial u} + 6\frac{\partial^3 f}{\partial x^2 \partial u_x}\right) = 0$
u_{xxxx}	$B_x\frac{\partial f}{\partial u_x} - B\left(2\frac{\partial f}{\partial u} + 3u_x\frac{\partial^2 f}{\partial u \partial u_x} + 3\frac{\partial^2 f}{\partial x \partial u_x}\right) = 0$

continued overleaf

$$u_{xx}u_{xxx} \qquad -10B\frac{\partial^2 f}{\partial u \partial u_x} = 0$$

$$u_{xx}u_{xxxx} \qquad -3B\frac{\partial^2 f}{\partial u_x^2} = 0$$

$$\text{Remaining} \quad -B_{xx}\left(2u_x\frac{\partial^2 f}{\partial x \partial u}+\frac{\partial^2 f}{\partial x^2}\right)-B_x\left(6u_x\frac{\partial^3 f}{\partial x^2 \partial u}+2\frac{\partial^3 f}{\partial x^3}\right)-B\left(4u_x\frac{\partial^4 f}{\partial x^3 \partial u}+\frac{\partial^4 f}{\partial x^4}\right)=0 \tag{11}$$

After some lengthy manipulations, the solution of the above set of equations is found to be:

$$f = f^1(x)u_x + f^2(x)u + f^3(x), \tag{12}$$

where:

$$f^1(x) = -2\iiint \frac{C_0x+C_1}{B}\,\mathrm{d}x^3 + \iint \frac{C_2-C_0x^2-2C_1x}{B}\,\mathrm{d}x^2 - C_6x^2+C_4x+C_5,$$

$$f^2(x) = \iint \frac{C_0x+C_1}{B}\,\mathrm{d}x^2 + C_6x + C_3,$$

$$f^3(x) = \iint \frac{C_7x+C_8}{B}\,\mathrm{d}x^2 + C_9x + C_{10}, \tag{13}$$

and all C_is are arbitrary constants.

Expressions (11) also impose the following constraint on the bending stiffness $B(x)$,

$$B_x\left(-2\iiint \frac{C_0x+C_1}{B}\,\mathrm{d}x^3 + \iint \frac{C_2-C_0x^2-2C_1x}{B}\,\mathrm{d}x^2 - C_6x^2+C_4x+C_5\right)$$
$$+B\left(4\iint \frac{C_0x+C_1}{B}\,\mathrm{d}x^2 - 3\int \frac{C_2-C_0x^2-2C_1x}{B}\,\mathrm{d}x + 4C_6x - 2C_3 - 3C_4\right) = 0. \tag{14}$$

Note that $\iiint \ldots \mathrm{d}x^3$ and $\iint \ldots \mathrm{d}x^2$ represent repeated integrations.

Having solved for the characteristic f, one can now proceed to construct divergence-free expressions, $D_xP^x = 0$ or $P^x = \text{constant}$, from the product $f\Delta$. With $\psi = u_x$ representing the rotation of the cross-section of the beam, $M = -B\psi_x$ denoting the bending moment, and $Q = M_x$ denoting the transverse shear force, the resulting conservation law is found to be:

$$P^x = -f^1(L+Qu_x-M\psi_x)+f^1_xM\psi+\tfrac{1}{2}f^1_{xx}Bu_x^2$$
$$+f^2(M\psi-Qu)+f^2_x(Mu+Bu_x^2)+f^2_{xx}(Buu_x-\tfrac{1}{2}B_xu^2)-\tfrac{1}{2}f^2_{xxx}Bu^2$$
$$-f^3Q+f^3_xM+f^3_{xx}(Bu_x-B_xu)-f^3_{xxx}Bu. \tag{15}$$

After substitution of f^1, f^2, and f^3 into the above equation, the conserved currents are given as:

$$\text{For only } C_0 \neq 0, \quad P^x = (L+Qu_x-M\psi_x)\left(2\iiint \frac{x}{B}\mathrm{d}x^3+\iint \frac{x^2}{B}\mathrm{d}x^2\right)$$
$$-M\psi\left(\iint \frac{x}{B}\mathrm{d}x^2+\int \frac{x^2}{B}\mathrm{d}x\right)-Qu\iint \frac{x}{B}\mathrm{d}x^2$$
$$+Mu\int \frac{x}{B}\mathrm{d}x+x\psi u-\tfrac{1}{2}x^2\psi^2-\tfrac{1}{2}u^2.$$

$$\text{For only } C_1 \neq 0, \quad P^x = (L+Qu_x-M\psi_x)\left(2\iiint \frac{1}{B}\mathrm{d}x^3+2\iint \frac{x}{B}\mathrm{d}x^2\right)$$
$$-M\psi\left(\iint \frac{1}{B}\mathrm{d}x^2+2\int \frac{x}{B}\mathrm{d}x\right)-Qu\iint \frac{1}{B}\mathrm{d}x^2$$
$$+Mu\int \frac{1}{B}\mathrm{d}x-x\psi^2+\psi u.$$

$$\text{For only } C_2 \neq 0, \quad P^x = -(L+Qu_x-M\psi_x)\iint \frac{1}{B}\mathrm{d}x^2+M\psi\int \frac{1}{B}\mathrm{d}x+\tfrac{1}{2}\psi^2.$$

$$\text{For only } C_3 \neq 0, \quad P^x = -Qu+M\psi.$$

$$\text{For only } C_4 \neq 0, \quad P^x = -x(L+Qu_x-M\psi_x)+M\psi.$$

$$\text{For only } C_5 \neq 0, \quad P^x = -(L+Qu_x-M\psi_x).$$

$$\text{For only } C_6 \neq 0, \quad P^x = x^2(L+Qu_x-M\psi_x)-x(Qu+M\psi)+Mu.$$

$$\text{For only } C_7 \neq 0, \quad P^x = -Q\iint \frac{x}{B}\mathrm{d}x^2+M\int \frac{x}{B}\mathrm{d}x+xu_x-u.$$

$$\text{For only } C_8 \neq 0, \quad P^x = -Q\iint \frac{1}{B}\mathrm{d}x^2+M\int \frac{1}{B}\mathrm{d}x+u_x.$$

$$\text{For only } C_9 \neq 0, \quad P^x = -xQ+M.$$

$$\text{For only } C_{10} \neq 0, \quad P^x = -Q. \tag{16}$$

It is important to note that for all the integrals appearing in the characteristic f, eqn (12), and in the conservation laws, eqn (16), integration constants are unnecessary since they can be absorbed in the constants C_3, C_4, C_5, C_6, C_9, and C_{10}.

To interpret the above conservation laws, it is necessary to introduce a distinction between physical space and material space. In physical space, conservation laws express physical balances of quantities such as forces, momenta and energy. These balance laws represent familiar concepts, such as force and moment equilibrium, and they can be obtained by considering free body diagrams in physical space. Conservation laws in material space express balances of quantities such as material force and wave momentum (for dynamics). These quantities are related to motions of material within the material space. The material force, which is a component in Eshelby's energy-momentum tensor (Eshelby, 1975) and termed material-momentum by Golebiewska-Herrmann (1981), can be regarded as a force on a defect and it is obtainable by considering a translation of the defect relative to the surrounding material. Similar to the connection between physical force and physical momentum, the rate of change of material force is identified as the wave momentum by Morse and Feshbach (1953) in the context of a vibrating string. To further understand the distinction and the significance of material and physical space, the readers can refer to publications by Golebiewska-Herrmann (1981) (1982a) (1982b) and (1983), where conservation laws in material space and their duality with those in physical space have been discussed extensively.

One feature of conservation laws in physical space is their independence of material properties. On inspection of the material constraint equation, eqn (14), it is apparent that the constants C_7, C_8, C_9 and C_{10} do not play any role in restricting the class of admissible nonhomogeneities. This implies that conservation laws corresponding to these constants are physical balance laws valid for any nonhomogeneity and each of them holds independently. In particular, the expression corresponding to $C_{10} \neq 0$ relates to the constancy of the shear force throughout the beam in the absence of a distributed transverse loading; the expression corresponding to $C_9 \neq 0$ embodies the relationship between the bending moment and the shear force; and the expressions corresponding to $C_7 \neq 0$ and $C_8 \neq 0$ show higher order balances between the bending moment and the shear force for a nonhomogeneous beam.

The remaining seven conserved quantities (C_0, C_1, ..., $C_6 \neq 0$) are divergence-free expressions in material space. Each of them expresses some balance of material force $(L+Qu_x-M\psi_x)$ of various orders. Of these seven laws, only $C_3 \neq 0$ does not contain a material force term. However, if one inspects eqn (14), one will observe that no material properties will satisfy the material constraint equation with only C_3 being nonzero, implying that this particular law only exists in combination with others.

Given any nonhomogeneity of the beam, one can construct divergence-free expressions in material space by solving the material constraint equation, eqn (14), for the relations among the C_is ($i = 0, 1, \ldots, 6$). These relations will provide the necessary combinations of the basic laws listed in expressions (16) to obtain conserved quantities. Conservation laws in material space will exist for the class of nonhomogeneities that satisfy the material constraint equation with at least one $C_i \neq 0$ ($i = 0, 1, \ldots, 6$). Two examples on construction of material conservation laws are given below.

For a homogeneous beam, where $B(x) = B_0$, the material constraint equation imposes the following constraints on the C_is ($i = 0, 1, \ldots, 6$),

$$\begin{aligned} C_0 &= C_1 = 0, \\ C_5 &\neq 0, \\ C_3 &= -\tfrac{3}{2}C_4, \\ C_2 &= \tfrac{4}{3}B_0C_6, \end{aligned} \tag{17}$$

and the associated conservation laws in material space are:

$$\begin{aligned} &\text{For } C_5 \neq 0, && P^x = -(L+Qu_x-M\psi_x). \\ &\text{For } C_3 = -\tfrac{3}{2}C_4, && P^x = -2x(L+Qu_x-M\psi_x)+3Qu-M\psi. \\ &\text{For } C_2 = \tfrac{4}{3}B_0C_6, && P^x = x^2(L+Qu_x-M\psi_x)+x(M\psi-3Qu)+2B_0\psi^2+3Mu. \end{aligned} \tag{18}$$

The first two laws in eqn (18) are identical to those obtained by Kienzler and Herrmann (1986a) which were derived by considering virtual displacements and material translation. They express the zeroth and first order balance of material force valid for a homogeneous beam. The third law in eqn (18), however, appears to be a new result and it expresses the second order balance of material force.

As a second example, consider a beam with bending stiffness $B(x) = B_0x^4$. In this case, eqn (14) requires that:

$$\begin{aligned} C_1 &= C_6 = 0, \\ C_2 &\neq 0, \\ C_4 &= 2C_3, \\ C_0 &= 4B_0C_5, \end{aligned} \tag{19}$$

with the corresponding conservation laws being:

$$\text{For } C_0 = 4B_0C_5,\quad P^x = -x^2(L+Qu_x-M\psi_x)-2[Mu+x(Qu-M\psi)] -2x^4B_0\psi^2+4x^3B_0\psi u-2x^2B_0u^2.$$

$$\text{For } C_4 = 2C_3,\quad P^x = 2x(L+Qu_x-M\psi_x)-Qu+3M\psi.$$

$$\text{For } C_2 \neq 0,\quad P^x = x^2(L+Qu_x-M\psi_x)-x(Qu+M\psi)+Mu. \tag{20}$$

Again, these three laws show first and second order balances of material force for the beam considered.

3.2. *Beams with distributed loading*

If one allows the presence of a transverse distributed load $q(x)$ acting on the beam, the Lagrangian density of the system will be modified to:

$$L = -\tfrac{1}{2}B(x)u_{xx}^2-q(x)u, \tag{21}$$

and the governing equation is given by:

$$\Delta = E(L) = B_{xx}u_{xx}+2B_xu_{xxx}+Bu_{xxxx}+q(x) = 0. \tag{22}$$

Following similar developments as in the case without loading, and assuming the characteristic to be of the form $f = f(x, u, u_x)$, the condition for the existence of conservation laws by the NA method, eqn (3), requires that:

$$f = f^1(x)u_x+f^2(x)u+f^3(x), \tag{23}$$

with $f^1(x)$ and $f^2(x)$ as given by eqn (13), and

$$f^3(x) = \iint \frac{1}{B}\left[\int f^1(x)q(x)\,dx-\iint f^2(x)q(x)\,dx^2\right]dx^2+\iint \frac{C_7x+C_8}{B}\,dx^2+C_9x+C_{10}, \tag{24}$$

where all C_is are arbitrary constants. The constraint on the bending stiffness $B(x)$ is identical to the case without loading as given by eqn (14).

The conserved current valid for this system is found to be:

$$P^x = -f^1(L+Qu_x-M\psi_x)+f^1_xM\psi+\tfrac{1}{2}f^1_{xx}Bu_x^2 +f^2(M\psi-Qu)+f^2_x(Mu+Bu_x^2)+f^2_{xx}(Buu_x-\tfrac{1}{2}B_xu^2)-\tfrac{1}{2}f^2_{xxx}Bu^2 +\int f^3(x)q(x)\,dx-f^3Q+f^3_xM+f^3_{xx}(Bu_x-B_xu)-f^3_{xxx}Bu. \tag{25}$$

At first glance, it may seem that the above conserved current is not a truly divergence-free expression due to the presence of an integral term. However, since both $f^3(x)$ and $q(x)$ are known functions of x, the term $\int f^3(x)q(x)\,dx$ can always be evaluated explicitly. To construct a divergence free expression for a beam under an arbitrary loading $q(x)$, we use eqns (13) and (24) to obtain the characteristic f, and construct the corresponding conservation law by eqn (25).

As a simple example, let us consider the case where $B(x) = B_0$ and $q(x) = q_0$. For this homogeneous beam under uniform loading, the material constraint equation (eqn 14) requires that:

$$
\begin{aligned}
C_0 = C_1 &= 0,\\
C_5 &\neq 0,\\
C_3 &= -\tfrac{3}{2}C_4,\\
C_2 &= \tfrac{4}{3}B_0C_6,
\end{aligned}
\tag{26}
$$

therefore, by eqns (13) and (24),

$$
\begin{aligned}
f^1(x) &= -\tfrac{1}{3}C_6x^2+C_4x+C_5,\\
f^2(x) &= C_6x-\tfrac{3}{2}C_4,\\
f^3(x) &= \frac{q_0}{B_0}(-\tfrac{1}{72}C_6x^5+\tfrac{5}{48}C_4x^4-\tfrac{1}{6}C_5x^3)+\frac{1}{6B_0}C_7x^3+\frac{1}{2B_0}C_8x^2+C_9x+C_{10}.
\end{aligned}
\tag{27}
$$

With $f^1(x)$, $f^2(x)$, and $f^3(x)$ known, conservation laws can be constructed directly via eqn (25), and the results are:

For $C_4 \neq 0$, $P^x = -x(L+Qu_x-M\psi_x)-\tfrac{1}{2}M\psi+\tfrac{3}{2}Qu$

$$+5\frac{q_0}{B_0}(\tfrac{1}{240}x^5q_0-\tfrac{1}{48}x^4Q+\tfrac{1}{12}x^3M+\tfrac{1}{4}x^2B_0u_x-\tfrac{1}{2}B_0xu).$$

For $C_5 \neq 0$, $P^x = -(L+Qu_x-M\psi_x)$.

For $C_6 \neq 0$, $P^x = \tfrac{1}{3}x^2(L+Qu_x-M\psi_x)+x(\tfrac{1}{3}M\psi-Qu)+\tfrac{2}{3}B_0u_x^2+Mu$

$$+\frac{q_0}{B_0}(-\tfrac{1}{432}x^6q_0+\tfrac{1}{72}x^5Q-\tfrac{5}{72}x^4M-\tfrac{5}{18}x^3B_0u_x+\tfrac{5}{6}x^2B_0u).$$

For $C_7 \neq 0$, $P^x = \tfrac{1}{24}x^4q_0-\tfrac{1}{6}x^3Q+\tfrac{1}{2}x^2M+xB_0u_x-B_0u.$

For $C_8 \neq 0$, $P^x = \tfrac{1}{6}x^3q_0-\tfrac{1}{2}x^2Q+xM+B_0u_x.$

For $C_9 \neq 0$, $P^x = \tfrac{1}{2}x^2q_0-xQ+M.$

For $C_{10} \neq 0$, $P^x = xq_0-Q.$ (28)

These conservation laws are valid for a homogeneous beam with a constant distributed loading, and each of them holds independently. The balance laws corresponding to C_5, C_4, $C_6 \neq 0$ are conservation laws in material space expressing the zeroth, first, and second order balance of material force, respectively; and those corresponding to C_7, C_8, C_9, $C_{10} \neq 0$ are laws in physical space showing various balances of bending moment, shear force and loading.

Given any general loading $q(x)$, and bending stiffness $B(x)$, corresponding conservation laws can be constructed as in the previous examples.

4. ELASTODYNAMICS OF BERNOULLI–EULER BEAMS

4.1. *Beams with end loading*

The Lagrangian density for dynamics of a nonhomogeneous Bernoulli–Euler beam is given by:

$$L = \tfrac{1}{2}H(x)u_t^2-\tfrac{1}{2}B(x)u_{xx}^2, \tag{29}$$

and the Euler–Lagrange equation that governs this system is:

$$\Delta = E(L) = Hu_{tt} + B_{xx}u_{xx} + 2B_x u_{xxx} + Bu_{xxxx} = 0. \tag{30}$$

Here, t is the time and $H(x)$ is the inertia term ρA.

Given the governing equation of the system, conservation laws valid for this system can be constructed in a manner similar to the statics case. Assuming the characteristic of conservation laws for this system to be

$$f = f^1(x,t)u_x + f^2(x,t)u_t + f^3(x,t)u + f^4(x,t), \tag{31}$$

the condition for the existence of conservation laws by the NA method as given by eqn (3) is:

$$E(f\Delta) \equiv 0, \tag{32}$$

which requires that:

$$\begin{aligned}
f^1 &= -2\iiint \frac{C_0x+C_1}{B}\,\mathrm{d}x^3 + \iint \frac{C_2 - C_0x^2 - 2C_1x}{B}\,\mathrm{d}x^2 - C_6x^2 + C_4x + C_5,\\
f^2 &= C_{11}t + C_{12},\\
f^3 &= \iint \frac{C_0x+C_1}{B}\,\mathrm{d}x^2 + C_6x + C_3,\\
f^4 &= \iint \frac{C_7x+C_8}{B}\,\mathrm{d}x^2 + C_{13}t + C_9x + C_{10},
\end{aligned} \tag{33}$$

where all C_is are arbitrary constants.

The condition for the existence of conservation laws also imposes the following constraints on the bending stiffness $B(x)$ and the inertia term $H(x)$:

$$\begin{aligned}
&B_x\left(-2\iiint \frac{C_0x+C_1}{B}\,\mathrm{d}x^3 + \iint \frac{C_2 - C_0x^2 - 2C_1x}{B}\,\mathrm{d}x^2 - C_6x^2 + C_4x + C_5\right)\\
&\quad + B\left(4\iint \frac{C_0x+C_1}{B}\,\mathrm{d}x^2 - 3\int \frac{C_2 - C_0x^2 - 2C_1x}{B}\,\mathrm{d}x + 4C_6x - 2C_3 - 3C_4 + C_{11}\right) = 0,\\
&H_x\left(-2\iiint \frac{C_0x+C_1}{B}\,\mathrm{d}x^3 + \iint \frac{C_2 - C_0x^2 - 2C_1x}{B}\,\mathrm{d}x^2 - C_6x^2 + C_4x + C_5\right)\\
&\quad + H\left(-4\iint \frac{C_0x+C_1}{B}\,\mathrm{d}x^2 + \int \frac{C_2 - C_0x^2 - 2C_1x}{B}\,\mathrm{d}x\right.\\
&\quad\quad \left. - 4C_6x - 2C_3 + C_4 - C_{11}\right) = 0.
\end{aligned} \tag{34}$$

Having solved for the characteristic f, one can proceed to construct divergence-free expressions, $D_xP^x + D_tP^t = 0$ from the product $f\Delta$. The resulting conservation law is found to be:

$$\begin{aligned}
P^x &= -f^1(L + Qu_x - M\psi_x) + f^1_xM\psi + \tfrac{1}{2}f^1_{xx}Bu_x^2 + f^2(M\psi_t - Qu_t)\\
&\quad + f^3(M\psi - Qu) + f^3_x(Mu + Bu_x^2) + f^3_{xx}(Buu_x - \tfrac{1}{2}B_xu^2) - \tfrac{1}{2}f^3_{xxx}Bu^2\\
&\quad - f^4Q + f^4_xM + f^4_{xx}(Bu_x - B_xu) - f^4_{xxx}Bu,\\
P^t &= f^1Hu_xu_t - f^2(L - Hu_t^2) + f^3Huu_t + f^4Hu_t - f^4_tHu.
\end{aligned} \tag{35}$$

After substitution of f^1, f^2, f^3 and f^4 into the above equations, the conserved currents can be written as:

For only $C_0 \neq 0$,
$$P^x = (L+Qu_x-M\psi_x)\left(2\iiint \frac{x}{B}\mathrm{d}x^3+\iint \frac{x^2}{B}\mathrm{d}x^2\right)$$
$$-M\psi\left(\iint \frac{x}{B}\mathrm{d}x^2+\int \frac{x^2}{B}\mathrm{d}x\right)-Qu\iint \frac{x}{B}\mathrm{d}x^2$$
$$+Mu\int \frac{x}{B}\mathrm{d}x+x\psi u-\tfrac{1}{2}x^2\psi^2-\tfrac{1}{2}u^2,$$
$$P^t = Hu_xu_t\left(-2\iiint \frac{x}{B}\mathrm{d}x^3-\iint \frac{x^2}{B}\mathrm{d}x^2\right)+Huu_t\iint \frac{x}{B}\mathrm{d}x^2.$$

For only $C_1 \neq 0$,
$$P^x = (L+Qu_x-M\psi x)\left(2\iiint \frac{1}{B}\mathrm{d}x^3+2\iint \frac{x}{B}\mathrm{d}x^2\right)$$
$$-M\psi\left(\iint \frac{1}{B}\mathrm{d}x^2+2\int \frac{x}{B}\mathrm{d}x\right)-Qu\iint \frac{1}{B}\mathrm{d}x^2$$
$$+Mu\int \frac{1}{B}\mathrm{d}x-x\psi^2+\psi u,$$
$$P^t = Hu_xu_t\left(-2\iiint \frac{1}{B}\mathrm{d}x^3-2\iint \frac{x}{B}\mathrm{d}x^2\right)+Huu_t\iint \frac{1}{B}\mathrm{d}x^2.$$

For only $C_2 \neq 0$,
$$P^x = -(L+Qu_x-M\psi_x)\iint \frac{1}{B}\mathrm{d}x^2+M\psi\int \frac{1}{B}\mathrm{d}x+\tfrac{1}{2}\psi^2,$$
$$P^t = Hu_xu_t\iint \frac{1}{B}\mathrm{d}x^2.$$

For only $C_3 \neq 0$,
$$P^x = -Qu+M\psi,$$
$$P^t = Huu_t.$$

For only $C_4 \neq 0$,
$$P^x = -x(L+Qu_x-M\psi_x)+M\psi,$$
$$P^t = xHu_xu_t.$$

For only $C_5 \neq 0$,
$$P^x = -(L+Qu_x-M\psi x),$$
$$P^t = Hu_xu_t.$$

For only $C_6 \neq 0$,
$$P^x = x^2(L+Qu_x-M\psi_x)-x(Qu+M\psi)+Mu,$$
$$P^t = -x^2Hu_xu_t+Huu_t.$$

For only $C_7 \neq 0$,
$$P^x = -Q\iint \frac{x}{B}\mathrm{d}x^2+M\int \frac{x}{B}\mathrm{d}x+xu_x-u,$$
$$P^t = Hu_t\iint \frac{x}{B}\mathrm{d}x^2.$$

For only $C_8 \neq 0$,
$$P^x = -Q\iint \frac{1}{B}\mathrm{d}x^2+M\int \frac{1}{B}\mathrm{d}x+u_x,$$
$$P^t = Hu_t\iint \frac{1}{B}\mathrm{d}x^2.$$

For only $C_9 \neq 0$, $P^x = -xQ+M,$

$P^t = xHu_t.$

For only $C_{10} \neq 0$, $P^x = -Q,$

$P^t = Hu_t.$

For only $C_{11} \neq 0$, $P^x = t(M\psi_t - Qu_t),$

$P^t = -t(L-Hu_t^2).$

For only $C_{12} \neq 0$, $P^x = M\psi_t - Qu_t,$

$P^t = -(L-Hu_t^2).$

For only $C_{13} \neq 0$, $P^x = -tQ,$

$$P^t = tHu_t - Hu. \tag{36}$$

Again, as in statics case, integration constants are unnecessary for all the integrals appearing in the characteristic f and in the conservation laws.

On inspection of eqn (34), which embodies the material constraint equations for the dynamics case, constants corresponding to C_is with $i = 7, 8, 9, 10, 12, 13$ do not play any role in restricting the admissible nonhomogeneities. Therefore, conservation laws corresponding to these C_is $\neq 0$ are physical balance laws valid for all material properties and each holds independently. With $i = 7, 8, 9, 10$ and 13 expressing various balances of bending moment, shear force and inertia; and $i = 12$ depicting the balance of total energy and the rate of work done valid for any nonhomogeneous beam.

The remaining eight divergence-free expressions are material balance laws for the dynamics of a nonhomogeneous Bernoulli–Euler beam. On inspection of eqn (34), the two laws that correspond to $C_3 \neq 0$ and $C_{11} \neq 0$ cannot exist independently. These two laws always exist in combination with the remaining six ($i = 0, 1, 2, 4, 5, 6$) which show various balances of material force $(L+Qu_x-M\psi_x)$ and wave momentum (Hu_xu_t).

Given any nonhomogeneous beam, material conservation laws that relate to the balance of material force and wave momentum can be constructed similarly to the static case. Using eqn (34) to establish the relation of the C_is that appear in this equation, material balance laws valid for that nonhomogeneous beam, if any, can be constructed with the basic expressions listed in eqn (36).

4.2. *Beams with distributed loading*

If one allows the presence of a transverse distributed load $q(x)$ acting on the beam, the Lagrangian density of the system will be modified to:

$$L = \tfrac{1}{2}H(x)u_t^2 - \tfrac{1}{2}B(x)u_{xx}^2 - q(x)u, \tag{37}$$

and the differential equation that governs this system is given by:

$$\Delta = E(L) = Hu_{tt} + B_{xx}u_{xx} + 2B_xu_{xxx} + Bu_{xxxx} + q(x) = 0. \tag{38}$$

Assuming the characteristic of conservation laws to be

$$f = f^1(x,t)u_x + f^2(x,t)u_t + f^3(x,t)u + f^4(x,t), \tag{39}$$

the condition for the existence of conservation laws by the NA method,

$$E(f\Delta) \equiv 0, \tag{40}$$

requires that:

$$f^1 = -2\iiint \frac{C_0x+C_1}{B}\,\mathrm{d}x^3 + \iint \frac{C_2-C_0x^2-2C_1x}{B}\,\mathrm{d}x^2 - C_6x^2+C_4x+C_5,$$

$$f^2 = C_{11}t+C_{12},$$

$$f^3 = \iint \frac{C_0x+C_1}{B}\,\mathrm{d}x^2+C_6x+C_3,$$

$$f^4 = \iint \frac{1}{B}\left[\int f^1 q\,\mathrm{d}x + \iint (f_t^2-f^3)q\,\mathrm{d}x^2\right]\mathrm{d}x^2 + \iint \frac{C_7x+C_8}{B}\,\mathrm{d}x^2+C_{13}t+C_9x+C_{10}, \tag{41}$$

where all C_is are arbitrary constants.

The constraints on the material properties, $B(x)$ and $H(x)$, are identical to the dynamic case without loading as given by eqn (34).

The corresponding conservation law is:

$$\begin{aligned} P^x = {} & -f^1(L+Qu_x-M\psi_x)+f_x^1M\psi+\tfrac{1}{2}f_{xx}^1Bu_x^2+f^2(M\psi_t-Qu_t) \\ & +f^3(M\psi-Qu)+f_x^3(Mu+Bu_x^2)+f_{xx}^3(Buu_x-\tfrac{1}{2}B_xu^2)-\tfrac{1}{2}f_{xxx}^3Bu^2 \\ & +\int f^4(x)q(x)\,\mathrm{d}x-f^4Q+f_x^4M+f_{xx}^4(Bu_x-B_xu)-f_{xxx}^4Bu, \\ P^t = {} & f^1Hu_xu_t-f^2(L-Hu_t^2)+f^3Huu_t+f^4Hu_t-f_t^4Hu. \end{aligned} \tag{42}$$

Given any $q(x)$, $B(x)$ and $H(x)$, conservation laws can be determined by first solving the material constraint equation, eqn (34), for the relation of the constants C_is, and then evaluating the precise form of f^is by eqn (41). Finally, conservation laws valid for that system can be obtained by eqn (42).

5. COMPARISON WITH NOETHER'S FIRST THEOREM

Conservation laws derived for the statics and dynamics of a non-homogeneous Bernoulli–Euler beam presented in this paper are obtained through the NA method proposed in Honein *et al.* (1991). Since a nonhomogeneous Bernoulli–Euler beam is a non-dissipative system which admits a variational formulation, the classical Noether's first theorem is also applicable for construction of conservation laws for this Lagrangian system. It is the purpose of this present section to discuss our results within the context of Noether's first theorem and to compare the two methods of constructing conservation laws.

Unlike the NA method, whose starting point for construction of divergence-free expressions is the governing equations of the system, Noether's first theorem starts with the Lagrangian density of the system. Thus, while the NA method is applicable to any system governed by a set of differential equations. Noether's methodology is applicable only to Lagrangian systems. Details on Noether's first theorem can be found in Noether (1918), Logan (1977), Olver (1986) and Bluman and Kumei (1989).

In brief, Noether stated that if a given system with m independent variables x^i ($i = 1, 2, \ldots, m$) and n dependent variables u^k ($k = 1, 2, \ldots, n$) is subjected to the set of infinitesimal transformations:

$$\begin{aligned} x^i \to x^{i*} &= x^i+\varepsilon\xi^i, \\ u^k \to u^{k*} &= u^k+\varepsilon\phi^k, \end{aligned} \tag{43}$$

where ε is an infinitesimal parameter, (ξ^i and ϕ^k are functions dependent on x^i and u^k for geometric symmetry, and also dependent on derivatives of u^k for generalized symmetry), the condition for the existence of conservation laws is that the action integral of the system be invariant under this set of transformations. With L being the Lagrangian density of the system, the condition for the existence of conservation laws is given by:

$$\int_{\Omega} L\,\mathrm{d}V = \int_{\Omega^*} L^*\,\mathrm{d}V^*. \tag{44}$$

Or explicitly,

$$(D_J Q^k)\frac{\partial L}{\partial u_J^k} + D_i(L\xi^i) = 0, \tag{45}$$

where

$$Q^k = \phi^k - \xi^i u_i^k. \tag{46}$$

Bessel-Hagen (1921) extended Noether's first theorem by inclusion of the so-called "divergence symmetries". Instead of requiring the strict invariance of the action integral under a set of infinitesimal transformations, Bessel-Hagen required that the action integral be invariant up to a divergence term. This requirement is stated mathematically as:

$$(D_J Q^k)\frac{\partial L}{\partial u_J^k} + D_i(L\xi^i) = D_i B^i, \tag{47}$$

where B^i is a set of arbitrary functionals. Thus, eqn (47) can be considered as the requirement for the existence of conservation laws for the general form of Noether's first theorem.

In Olver (1986), it is shown that Noether's condition for the existence of conservation laws, eqn (47), implies:

$$D_i P^i = Q^k E^k(L), \tag{48}$$

where $E^k(L)$ designates the Euler–Lagrange equations governing the system, and Q^k are the characteristics of conservation laws within the framework of Noether. On inspection of eqn (48), the parallel between Noether's and the NA method of constructing conservation laws becomes immediately apparent. In the NA method, one seeks the form of the characteristic f^k such that divergence-free expressions can be constructed out of the product of the characteristics and the governing equations Δ^k, or explicitly:

$$D_i P^i = f^k \Delta^k. \tag{49}$$

For Lagrangian systems, the governing equations are the Euler–Lagrange equations of the system:

$$\Delta^k = E^k(L). \tag{50}$$

In such a case, the characteristics (Q^k) within Noether's framework and the characteristics (f^k) in the NA method coincide.

In fact, it has been shown in detail by Chien (1992) that for Lagrangian systems the requirement for the existence of conservation laws by the NA method and by Noether's first theorem as extended by Bessel-Hagen are mathematically identical. All conservation laws obtainable through Noether's first theorem can also be constructed via the NA method. Even though the two methodologies for construction conservation laws are essentially equivalent when applied to Lagrangian systems, it will be shown here via the following example that in some cases the NA method might be more efficient.

For the case of a homogeneous beam with end loading only, the conservation law corresponding to $C_2 = \frac{4}{3}B_0C_6$ in eqn (18) is given as:

$$P^x = x^2(L+Qu_x - M\psi_x) + x(M\psi - 3Qu) + 2B_0\psi^2 + 3Mu. \tag{51}$$

To obtain this balance law within Noether's framework, we must use Noether's first theorem with the extension by Bessel-Hagen including divergence symmetries, which corresponds to eqn (47), requiring that:

$$\begin{aligned} \xi &= x^2, \\ \phi &= 3xu, \\ B^x &= -2Bu_x^2. \end{aligned} \tag{52}$$

For this special case, the NA method of constructing conservation laws is shown to be more efficient than the classical method of Noether's. While Noether's methodology requires the use of three unknown functions ($\xi = \xi(x, u)$, $\phi = \phi(x, u)$, $B^x = B^x(x, u, u_x)$), the NA method uses only one unknown function ($f = f(x, u, u_x)$) to arrive at the same conservation law.

6. CONCLUSIONS

In the present contribution, conservation laws obtained through the NA method of constructing divergence-free expressions are presented for the statics and dynamics of a nonhomogeneous Bernoulli–Euler beam. Two sets of conservation laws are obtained. The first set consists of conservation laws in physical space, which express various balances of bending moment, shear force, loading, energy and rate of work done. These physical balance laws are valid for any nonhomogeneous beam. The other set consists of conservation laws in material space, which express various balances of material force and wave momentum. Material balance laws are found to be valid only for a certain class of nonhomogeneities. If a given nonhomogeneity of the beam satisfies the material constraint equations with no null results on the constants appearing in the constraint equations, then divergence-free expressions in material space exist for that class of material properties.

A complete duality between physical balance laws and material balance laws is established in Golebiewska-Herrmann (1981) (1982a) (1982b) and (1983), where quantities such as physical stress and material-momentum are placed on equal footing describing the properties of physical and material space, respectively. As physical balance laws describe the motion of the system in physical space, material balance laws as discussed in the above references can be used to describe the motion of defects in material space. Consequently, the material conservation laws derived here might be useful in the analysis of fracture and defects of nonhomogeneous beams.

Since a nonhomogeneous Bernoulli–Euler beam is a Lagrangian system, conservation laws can be derived for this system using the classical Noether's first theorem. As has been shown by Chien (1992), the NA method and Noether's theorem for constructing conservation laws yield identical results for Lagrangian systems, all conservation laws presented here can be constructed using Noether's first theorem as extended by Bessel-Hagen. However, on comparison of the two methodologies, the NA method seems to be more powerful. While Noether's first theorem involving generalized symmetry provides the theoretical groundwork for obtaining divergence-free expressions, it offers no systematic procedure for doing so. The NA method, however, allows its users a systematic approach. Also, it is shown by a specific example here that the NA method for constructing conservation law might be more efficient than the classical Noether's first theorem. Thus, aside from being applicable to dissipative systems, the NA method should be more advantageous than Noether's theorem in the construction of conservation laws even for Lagrangian systems.

The conservation laws obtained here are not exhaustive. If one allows the dependence of the characteristic to include higher order derivatives of the dependent variable, additional conservation laws may be obtained. For this Lagrangian (self-adjoint) system governed by a linear differential equation higher order conservation laws that are quadratic in u and its derivatives can be obtained using a linear recursive operator as discussed in Olver (1986). This type of conservation law corresponds to characteristics that are linear in u and its derivatives. Using a recursive operator, conservation laws that depend quadratically on higher order derivatives of u can be constructed from lower order dependence quadratic conservation laws. It can be shown that the constraint on the bending stiffness function $B(x)$ for higher order conservation laws obtained by recursion is identical to that of the original lower order conservation law.

As is noted in Olver (1986), a self-adjoint linear system with one quadratic conservation law always has an infinite hierarchy of such laws depending on higher and higher order derivatives of u. Once we employ a recursive operator, we must also investigate topics such as equivalent conservation laws, trivial conservation laws and completeness.

To obtain conservation laws which correspond to characteristics which are quadratic or of higher order in u and its derivatives, one must solve for general nonlinear characteristics. This extension, however, is not possible at the moment due to difficulties in obtaining the explicit form of the existence condition for conservation laws with a higher order nonlinear dependence characteristic. It is hoped that this difficulty will be overcome in the near future with the aid of symbolic manipulation on computing devices with large memory.

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REFERENCES

Bessel-Hagen, E. (1921). Über die Erhaltungssätze der Electrodynamik. *Math. Ann.* **84**, 258–276.

Bluman, G. W. and Kumei, S. (1989). *Symmetries and Differential Equations.* Springer-Verlag, New York.

Chien, N. Y. (1992). Conservation laws in non-homogeneous and dissipative mechanical systems, Ph.D. Dissertation, Stanford University.

Eshelby, J. D. (1975). The elastic energy–momentum tensor. *J. Elast.* **5**, 321–335.

Golebiewska-Herrmann, A. (1981). On conservation laws of continuum mechanics. *Int. J. Solids Structures* **17**, 1–9.

Golebiewska-Herrmann, A. (1982a). On physical and material conservation laws. *Proc. IUTAM Symp. Finite Elasticity at Lehigh University.* Martinus Nijhoff, Boston, 201–209.

Golebiewska-Herrmann, A. (1982b). Material momentum tensor and path-independent integrals of fracture mechanics. *Int. J. Solids Structures* **18**, 319–326.

Golebiewska-Herrmann, A. (1983). On the Lagrangian formulation of continuum mechanics. *Physica* **118A**, 300–314.

Honein, T., Chien, N. and Herrmann, G. (1991). On conservation laws for dissipative systems. *Phys. Lett. A.* **155**, 223–224.

Kienzler, R. and Herrmann, G. (1986a). On material forces in elementary beam theory. *J. Appl. Mech.* **53**, 561–564.

Kienzler, R. and Herrmann, G. (1986b). An elementary theory of defective beams. *Acta Mechanica* **62**, 37–46.

Logan, J. D. (1977). *Invariant Variational Principles.* Academic Press, New York.

Morse, P. M. and Feshbach, H. (1953). *Methods of Theoretical Physics.* McGraw-Hill, New York.

Noether, E. (1918). Invariante Variationsprobleme, *Nachr. K. Ges. Wiss. Göttingen, Math.-Phys. Kl.* 235–257 (see *Transport Theory Stat. Phys.* **1** (1971), 186–207 for an English translation).

Olver, P. J. (1986). *Applications of Lie Groups to Differential Equations.* Springer, New York.